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THE NORMALIZED CYCLOMATIC QUOTIENT ASSOCIATED WITH PRESENTATIONS OF FINITELY GENERATED GROUPS

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ABSTRACT

Given a presentation of an n-generated group, we define the normalized cyclomatic quotient (NCQ) of it, which gives a number between $1-n$ and 1. It is computed through an investigation of the asymptotic behavior of a kind of an "average rank", or more precisely the quotient of the rank of the fundamental group of a finite subgraph of the corresponding Cayley graph by the size of the subgraph. In many ways (but not always) the NCQ behaves similarly to the behavior of the spectral radius of a symmetric random walk on the graph. In particular, it characterizes amenable groups. For some types of groups, like finite, amenable or free groups, its value equals that of the Euler characteristic of the group. We give bounds for the value of the NCQ for factor groups and subgroups, and formulas for its value on direct and free products. Some other asymptotic invariants are also discussed.

1. Introduction

The present paper is about an asymptotic invariant of presentations of groups, which is an invariant of the group when it is (finite or infinite) amenable. In fact, this invariant, which we call the normalized cyclomatic quotient (NCQ) and denote by $\hat{\Xi}$, characterizes amenable groups, the same as does the spectral radius

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of a symmetric random walk on the graph of the presentation (see [8], and also [4] for the connection between the spectral radius and the "growth-exponent"). Moreover, some of the formulas and bounds we get for the value of the NCQ on different group structures are similar to those valid for the spectral radius, but this is not always the case (see e.g. the remark after Theorem 3.1). Although the computation of the exact value of the NCQ is not easy and seems to be difficult for many kinds of presentations of groups (but good lower and upper bounds are sometimes easier to achieve), in some cases it may be easier than the computation of the spectral radius. For example, unlike the situation with the spectral radius (as far as we know), one does have a formula for the value of the NCQ on free products (see section 5).

Given the Cayley graph of a finitely generated group G, with respect to a presentation G^α with n generators, the quotient of the rank of the fundamental group of subgraphs of the Cayley graph by the cardinality of the set of vertices of the subgraphs (i.e. a kind of an "average rank") gives rise to the definition of the normalized cyclomatic quotient $\hat{\Xi}(G^\alpha)$. The asymptotic behavior of this quotient is similar to the asymptotic behavior of the quotient of the cardinality of the boundary of the subgraph by the cardinality of the subgraph. Using Følner's criterion for amenability one gets that the NCQ vanishes for infinite groups if and only if they are amenable. When G is finite then $\hat{\Xi}(G^\alpha) = 1/|G|$, where $|G|$ is the cardinality of G, and when G is non-amenable then $1 - n \leq \hat{\Xi}(G^\alpha) < 0$, with $\hat{\Xi}(G^\alpha) = 1 - n$ if and only if G is free of rank n. Thus we see that on special cases $\hat{\Xi}(G^\alpha)$ equals the Euler characteristic of G (see [1]), but this is not the case in general, as can be shown e.g. for free products with amalgamation of two free groups.

As said above, the value of the NCQ depends on the presentation of the group (unless the group is amenable). In the next section we suggest a definition of an invariant of the group itself, but this invariant seems to be difficult to compute in general.

Most of the paper is concerned with bounds for the value of the NCQ on factor groups and subgroups, and formulas with respect to the decomposition of the group into direct and free products. In the last section we define and touch very briefly the balanced cyclomatic quotient, which is defined on concentric balls in the graph. This definition is related to the growth of G.

Throughout the paper we assume that when we are given presentations $G_i^{\alpha_i}$

of groups G_i and H is a group defined through the G_i, then H gets the natural **induced** presentation H^α. Thus, if H is a factor of G then H^α has the same generating set as that of the presentation G^α of G with all the relations of G^α holding also in H^α. Or if $H = G_1 * G_2$ then the generators and relators of H^α are the union of those of the $G_i^{\alpha_i}$, assuming that the generating sets of the $G_i^{\alpha_i}$ are disjoint. Similarly for direct products (with the appropriate commutators as extra relators), etc. We do not try, however, to be too precise with regard to the use we make of the notation G^α.

The formulas we give for direct and free products serve as upper and lower bounds for $\hat{\Xi}(G^\alpha)$ in the following sense. Suppose that $G_i^{\alpha_i} = \langle X_i |\, R_i\rangle$, $i = 1, 2$, and $G^\alpha = \langle X |\, R\rangle$, with $X = X_1 \cup X_2$ (disjoint union), $R \supseteq R_1 \cup R_2$ and so that the natural maps $G_i \to G$, $i = 1, 2$, are injective. Then

$$\hat{\Xi}(G_1^{\alpha_1} * G_2^{\alpha_2}) \le \hat{\Xi}(G^\alpha) \le \hat{\Xi}(G_1^{\alpha_1} \times G_2^{\alpha_2}). \tag{1}$$

The left inequality follows from Theorem 3.1 since the Cayley graph associated with G^α is a quotient of that associated with $G_1^{\alpha_1} * G_2^{\alpha_2}$. The right inequality appears in the proof of Theorem 4.1. These bounds hold for such structures as semi-direct products, amalgamated products, HNN extensions, etc. One can also try and get exact formulas for the structures mentioned above, at least in special cases, but we do not get much into it in the present paper.

We hope that the results as well as the methods of computation (e.g. building the subgraphs inductively from smaller subgraphs on which we know more about the number of cycles and vertices, often using translates of the same smaller subgraph, or looking at quotient graphs, or counting cycles, edges or outer edges of subgraphs, according to what is convenient, etc.) will turn out to have interesting applications. For example, we can obtain the generating function of the growth of a surface group of genus n (which we were told is a known result), by adding inductively the basic subgraphs which are cycles of length $4n$, and observing there are 2 types of these cycles according to whether we add $2n-2$ or $2n-3$ new vertices (we do not have the space here to give the exact details). Another application is a corollary to Theorem 3.1 which is a theorem of Lubotzky and Weiss ([10]) about non-expander groups (see the remark after Theorem 3.1).

We use the following terminology and notation on graphs. The set of vertices of a graph $\mathcal{G}$ is denoted by $V(\mathcal{G})$ and the set of edges by $E(\mathcal{G})$. A **path** in $\mathcal{G}$ is a sequence $v_0, e_1, v_1, e_2, \ldots, v_k \in V(\mathcal{G})$, $e_i \in E(\mathcal{G})$ such that e_i starts at the vertex v_{i-1} and terminates at v_i. The length of a path $v_0, e_1, v_1, e_2, \ldots, v_n$ is n. A **simple path** is a path in which the vertices along it are distinct, except possibly for the first and last one, in which case it is a **simple closed path** or a **simple circuit**. We assume that each path is **reduced**, i.e. it is not homotopic to a shorter one when the initial and terminal vertices are kept fixed.

If $\mathcal{H} \subseteq \mathcal{G}$, i.e. $\mathcal{H}$ is a collection of vertices and edges of the graph $\mathcal{G}$, then we denote by $\langle\mathcal{H}\rangle$ the subgraph **generated** by $\mathcal{H}$. It is the smallest subgraph of $\mathcal{G}$ which contains $\mathcal{H}$. That is, we add to $\mathcal{H}$ the endpoint vertices of all the edges in $\mathcal{H}$. On the other hand, the subgraph of $\mathcal{G}$ **induced** by $\mathcal{H}$ is the one whose vertices are those of $\mathcal{H}$ and whose edges are all the edges which join these vertices in $\mathcal{G}$. An induced subgraph is a subgraph which is induced by some $\mathcal{H} \subseteq \mathcal{G}$. If $\mathcal{H}_1, \mathcal{H}_2 \subseteq \mathcal{G}$ then $\mathcal{H}_1 - \mathcal{H}_2$ is the collection of vertices $V(\mathcal{H}_1) - V(\mathcal{H}_2)$ and edges $E(\mathcal{H}_1) - E(\mathcal{H}_2)$, and it does not necessarily form a subgraph of $\mathcal{G}$, even when $\mathcal{H}_1$ and $\mathcal{H}_2$ are subgraphs of $\mathcal{G}$. The **boundary** of the subgraph $\mathcal{H}$ of $\mathcal{G}$ is $\partial\mathcal{H} = \mathcal{H} \cap \langle\mathcal{G} - \mathcal{H}\rangle$, and its **interior** is $\dot{\mathcal{H}} = \mathcal{H} - \partial\mathcal{H}$. The **outer boundary** of $\mathcal{H}$ (in $\mathcal{G}$) is the set of vertices of $\mathcal{G} - \mathcal{H}$ which are adjacent to $\mathcal{H}$ in $\mathcal{G}$. Assume now that each edge of $\mathcal{G}$ is labeled with some $x \in X$ in one direction and with $x^{-1} \in X^{-1}$ in the other direction. Then we define $E^X_{out}(\mathcal{H})$ to be the set of edges of $\mathcal{G} - \mathcal{H}$ whose initial vertices with respect to the directions X are in $\mathcal{H}$.

Finally, let $\beta_0(\mathcal{H}) = |V(\mathcal{H})|$, let $\beta_1(\mathcal{H}) = |E(\mathcal{H})|$, and let $\alpha(\mathcal{H}) = |\pi_0(\mathcal{H})|$ be the number of the (connected) components of $\mathcal{H}$.

2. The normalized cyclomatic quotient

Let $G^\alpha = \langle X | R\rangle$ be a presentation of a group G, with $X = \{x_1, \ldots, x_n\}$, and let $\mathcal{G}$ be the associated Cayley graph. If H is the normal closure of R in $F = \langle X\rangle$ then it is shown in [13] that the growth function of G is equivalent to the **rank-growth** rk_H of H. The rank-growth is defined by

$$rk_H(i) = \text{rank}(H_i), \tag{2}$$

where H_i is the subgroup of H generated by the elements of length $\leq i$ (with respect to $X \cup X^{-1}$). We notice that H_i is the fundamental group of the subgraph of $\mathcal{G}$ of all paths starting at 1 of length $\leq i$. Thus, there exists an **exhausting chain** $(\mathcal{G}'_i)$ in $\mathcal{G}$, i.e. a sequence $\mathcal{G}'_1 \subseteq \mathcal{G}'_2 \subseteq \mathcal{G}'_3 \subseteq \cdots$ of subgraphs of $\mathcal{G}$ whose union is $\mathcal{G}$, with all the $\mathcal{G}'_i$ connected and finite, such that the growth of the function $\gamma_1(i) = \text{rank}(\pi_1(\mathcal{G}'_i))$ is equivalent to the growth of the function $\gamma_2(i) = |V(\mathcal{G}'_i)|$.

We are now interested in the asymptotic behavior of the quotient $\gamma_1(i)/\gamma_2(i)$, which is related, as we will see, to the quotient $|V(\partial\mathcal{G}_i')|/|V(\mathcal{G}_i')|$. The latter quotient and its analogs are known and widely studied objects in diverse areas of Mathematics (see e.g. the survey [9]). By Følner's criterion (see [6]) the group G is amenable if and only if there exists an exhausting chain $(\mathcal{G}_i')$ of finite connected subgraphs of $\mathcal{G}$ such that

$$\limsup_{i\to\infty}\frac{\beta_0(\partial\mathcal{G}_i)}{\beta_0(\mathcal{G}_i)}=0 \tag{3}$$

(recall that a group G is amenable if there exists an invariant mean on $B(G)$, the space of all bounded complex-valued functions on G with the sup norm $\| f \|_\infty$, see [3]). We remark that in Følner's criterion one can consider as well disconnected subgraphs or boundaries of any fixed width k. Notice that (3) implies that if G is non-amenable then it has exponential growth.

Let us denote by $\beta_2(\mathcal{G}')$ the **cyclomatic number** of $\mathcal{G}'$, i.e. the sum of the values of $\operatorname{rank}(\pi_1(\mathcal{H}'))$ over all the components $\mathcal{H}'$ of the subgraph $\mathcal{G}'$ (the notation $\beta_2(\mathcal{G}')$ refers to the number of 2-cells of an associated 2-dimensional complex). Let us also use the following notation:

$$\xi(\mathcal{G}') = \frac{\beta_2(\mathcal{G}')}{\beta_0(\mathcal{G}')}, \tag{4}$$

$$\mu(\mathcal{G}') = \frac{\beta_1(\mathcal{G}')+1}{\beta_0(\mathcal{G}')}. \tag{5}$$

We denote by $\mathcal{F}(\mathcal{G}), \mathcal{F}^*(\mathcal{G}). \mathcal{CF}(\mathcal{G}), \mathcal{CF}^*(\mathcal{G})$ respectively the sets of finite, non-trivial finite, connected finite, non-trivial connected finite subgraphs of $\mathcal{G}$. Here a graph is non-trivial if it contains more than one vertex.

Definition 2.1: If $C=(\mathcal{G}_1\subseteq\mathcal{G}_2\subseteq\cdots)$, $\mathcal{G}_i\in\mathcal{CF}(\mathcal{G})$, is an exhausting chain, let

$$\eta_C=\limsup_{i\to\infty}\xi(\mathcal{G}_i). \tag{6}$$

Then we define the **cyclomatic quotient** of G^α by

$$\Xi(G^\alpha)=\sup_C\eta_C, \quad C \text{ an exhausting chain,} \tag{7}$$

and the **normalized cyclomatic quotient** of G^α by

$$\hat{\Xi}(G^\alpha)=1-n+\Xi(G^\alpha). \tag{8}$$

The following are equivalent definitions of $\hat{\Xi}(G^\alpha)$.

$$\hat{\Xi}(G^\alpha) = 1 - n + \sup_{\mathcal{G}' \in \mathcal{F}(\mathcal{G})} \xi(\mathcal{G}'), \tag{9}$$

$$\hat{\Xi}(G^\alpha) = \sup_{\mathcal{G}' \in C\mathcal{F}(\mathcal{G})} \frac{1 - |E^X_{out}(\mathcal{G}')|}{\beta_0(\mathcal{G}')}, \tag{10}$$

$$\hat{\Xi}(G^\alpha) = -n + \sup_{\mathcal{G}' \in C\mathcal{F}(\mathcal{G})} \mu(\mathcal{G}'), \tag{11}$$

$$\hat{\Xi}(G^\alpha) = \sup_S \left(\frac{1}{|S|} + \sum_{j=1}^{n} \left(\frac{|Sx_j \cap S|}{|S|} - 1 \right) \right), \quad S \subseteq G \text{ finite}. \tag{12}$$

In case G is infinite then we have

$$\begin{aligned} &\hat{\Xi}(G^\alpha) = -\inf_{\mathcal{G}' \in C\mathcal{F}(\mathcal{G})} \tfrac{|E^X_{out}(\mathcal{G}')|}{\beta_0(\mathcal{G}')}, \\ &\hat{\Xi}(G^\alpha) = -n + \sup_{\mathcal{G}' \in C\mathcal{F}(\mathcal{G})} \tfrac{\beta_1(\mathcal{G}')}{\beta_0(\mathcal{G}')}, \\ &\hat{\Xi}(G^\alpha) = \sup_S \textstyle\sum_{j=1}^{n} \left(\tfrac{|Sx_j \cap S|}{|S|} - 1 \right), \quad S \subseteq G \text{ finite}. \end{aligned} \tag{13}$$

Clearly definition (9) gives at least the same value as definition (8). To see that these definitions are equivalent we need to show that for every $\mathcal{G}', \mathcal{G}'' \in \mathcal{F}(\mathcal{G})$ there exists $\mathcal{H} \in C\mathcal{F}(\mathcal{G})$ such that $\mathcal{G}'' \subseteq \mathcal{H}$ and $\xi(\mathcal{G}') \leq \xi(\mathcal{H})$. But this follows by Lemma 2.2 and by the fact that given any subgraphs $\mathcal{G}_1, \mathcal{G}_2 \in \mathcal{F}(\mathcal{G})$, we can cover $\mathcal{G}_2$ by the induced subgraph of translates of $\mathcal{G}_1$.

LEMMA 2.2: *Let $\mathcal{G}' \in \mathcal{F}(\mathcal{G})$ satisfy $\xi(\mathcal{G}') \geq \xi(\mathcal{H}')$ for every $\mathcal{H}' \subseteq \mathcal{G}'$. Let $\mathcal{G}'' \in \mathcal{F}(\mathcal{G})$ such that $\xi(\mathcal{G}'') \geq \xi(\mathcal{G}')$, and let $\mathcal{H} = \mathcal{G}' \cup \mathcal{G}''$. Then $\xi(\mathcal{H}) \geq \xi(\mathcal{G}')$.*

Proof: Clearly

$$\beta_0(\mathcal{H}) = \beta_0(\mathcal{G}') + \beta_0(\mathcal{G}'') - \beta_0(\mathcal{G}' \cap \mathcal{G}''), \tag{14}$$

while

$$\beta_2(\mathcal{H}) \geq \beta_2(\mathcal{G}') + \beta_2(\mathcal{G}'') - \beta_2(\mathcal{G}' \cap \mathcal{G}''). \tag{15}$$

Then by simple calculation we get that

$$\xi(\mathcal{H}) \geq \xi(\mathcal{G}') \tag{16}$$

after clearing denominators in the quotients $\xi(\mathcal{H})$ and $\xi(\mathcal{G}')$. ∎

Definition (10) follows by

$$\beta_2(\mathcal{G}') = 1 + (n-1)\beta_0(\mathcal{G}') - |E^X_{out}(\mathcal{G}')|, \quad \mathcal{G}' \in C\mathcal{F}(\mathcal{G}). \tag{17}$$

Definition (11) follows by

$$\beta_2(\mathcal{G}') = \alpha(\mathcal{G}') - \beta_0(\mathcal{G}') + \beta_1(\mathcal{G}') \tag{18}$$

($\alpha(\mathcal{G}')$ is the number of components of $\mathcal{G}'$).

Finally, for definition (12) we notice that if $\mathcal{G}'$ is an induced subgraph of $\mathcal{G}$ and that $S \subseteq G$ is the set of elements of G which correspond to the vertices of $\mathcal{G}'$ then

$$\beta_1(\mathcal{G}') = \sum_{j=1}^{n} |Sx_j \cap S|. \tag{19}$$

The slightly simplified expressions (13) when G is infinite are due to the fact that $\mathcal{G}'$ may be chosen as large as we wish.

Now before we investigate $\hat{\Xi}(G^\alpha)$, we would like to add few remarks about its definition. First, we could have used the expressions (13) as defining $\hat{\Xi}(G^\alpha)$. Then, as we will see in the next sections, the formulas for some of the structures would have been simpler. The reason for our choice is that we wanted to leave the part in the definition that makes the distinction between finite groups of different order, keeping in mind that when the group is infinite the two definitions agree with each other. Secondly, we have defined an invariant of a presentation of a group and not of the group itself (and we will see below some examples of the effect of changing the presentation of the same group on the value of $\hat{\Xi}$). We may, however, define $\hat{\Xi}(G)$ by $\hat{\Xi}(G) = \sup_{G^\alpha} \hat{\Xi}(G^\alpha)$, where G^α runs over all the (isomorphism classes) of the presentations of G with finitely many generators (that is, we may assume an infinite sequence $X = \{x_1, x_2, \ldots\}$ is given and for each n the n-generated presentations are with $x_1, \ldots, x_n$ as generators). The problem with this definition that it seems to be difficult to compute in general, although by Proposition 2.4 it suffices to consider only "minimal" presentations. Interesting questions would then be whether $\hat{\Xi}(G) = \hat{\Xi}(G^\alpha)$ for some presentation G^α of G, or whether it may happen that $\hat{\Xi}(G) = 0$ for some non-amenable group. Thirdly, $\hat{\Xi}(G^\alpha)$ may be defined also for groups presented by countably many generators $X = \{x_1, x_2, \ldots\}$. Let $G_i = Gp(x_1, x_2, \ldots, x_i)$ be the subgroup of G generated by $\{x_1, x_2, \ldots, x_i\}$. Then define $\hat{\Xi}(G^\alpha) = \lim_{i \to \infty} \hat{\Xi}(G_i^\alpha)$. This sequence is monotonous non-increasing and thus has a limit in the broad sense. In fact, the order of the generators does not affect the value of $\hat{\Xi}(G^\alpha)$.

PROPOSITION 2.3: *Let G^α be a presentation of a group G with n generators and let $\mathcal{G}$ be the associated Cayley graph. Then $1 - n \leq \hat{\Xi}(G^\alpha) \leq 1$. Moreover,*

(i) *if G is amenable then* $\hat{\Xi}(G^\alpha) = 1/|G|$, *where* $1/|G|$ *is defined to be* 0 *if* $|G| = \infty$;

(ii) *if G is non-amenable then* $1 - n \leq \hat{\Xi}(G^\alpha) < 0$, *with* $\hat{\Xi}(G^\alpha) = 1 - n$ *if and only if G is free of rank* $n \geq 2$.

Proof: If G is finite then any exhausting chain of $\mathcal{G}$ stabilizes on $\mathcal{G}$. Then by (10)

$$\hat{\Xi}(G^\alpha) = \frac{1 - |E^X_{out}(\mathcal{G})|}{\beta_0(\mathcal{G})} = \frac{1}{|G|}. \tag{20}$$

In fact, we see that for every proper subgraph $\mathcal{G}'$ of $\mathcal{G}$

$$\frac{1 - |E^X_{out}(\mathcal{G}')|}{\beta_0(\mathcal{G}')} \leq 0. \tag{21}$$

Since for every $\mathcal{G}' \in \mathcal{F}(\mathcal{G})$, $|E^X_{out}(\mathcal{G}')|$ is of the same order as $\beta_0(\partial\mathcal{G}')$, then by using Følner's criterion we obtain from (10) that $\hat{\Xi}(G^\alpha) = 0$ when G is infinite amenable.

When G is non-amenable then there exists $c > 0$ such that for every $\mathcal{G}' \in \mathcal{F}(\mathcal{G})$, $|E^X_{out}(\mathcal{G}')|/\beta_0(\mathcal{G}') > c$. Then by (10) $\hat{\Xi}(G^\alpha) \leq -c$, by letting $\beta_0(\mathcal{G}') \to \infty$. On the other hand, $\hat{\Xi}(G^\alpha) > 1 - n$ when there exits at least one circuit in $\mathcal{G}$. When $\mathcal{G}$ contains no circuits then G is free of rank n and $\hat{\Xi}(G^\alpha) = 1 - n$. ∎

If we use (11) for the definition of $\hat{\Xi}(G^\alpha)$ then by Proposition 2.3 we get the following criterion for amenability: G is amenable if and only if for every $\epsilon > 0$ there exists $\mathcal{G}' \in \mathcal{F}(\mathcal{G})$ such that

$$\frac{\beta_1(\mathcal{G}')}{\beta_0(\mathcal{G}')} > n - \epsilon. \tag{22}$$

In other words, G is amenable if and only if for every $\epsilon > 0$ there is a finite subset S of G such that $|Sx_j \cap S|/|S| > 1 - \epsilon$ for each generator x_j. This is easily seen to be equivalent to Følner's criterion for amenability ([2]): for every $\epsilon > 0$ and every $w_1, \ldots, w_r \in G$ there is a finite subset S of G such that $|Sw_i \cap S|/|S| > 1 - \epsilon$ for each i. (This also shows that a subgroup of an amenable group is amenable.)

The value of $\hat{\Xi}(G^\alpha)$ depends on the presentation G^α of G. Let us look at the following example. Let G^{α_1} be the presentation of the free group G of rank 2 with generators x, y. Then $\hat{\Xi}(G^{\alpha_1}) = 1 - 2 = -1$. Let G^{α_2} be obtained from G^{α_1} by the Tietze transformation of adding a new generator x' and a relation $x' = w$. If $w = x^k$ for some integer k then $\hat{\Xi}(G^{\alpha_2}) = \hat{\Xi}(G^{\alpha_1})$, as can be seen from the chain of subgraphs whose vertices consist of increasing powers of x. On the other hand, if $w = xy$ then the only simple circuits we get are triangles of the form $x'y^{-1}x^{-1}$ (or cyclic permutations of it) and by forming an increasing chain of subgraphs the best we can do is adding each time 2 new vertices and obtaining a new circuit. Therefore we get that $\hat{\Xi}(G^{\alpha_2}) = (1-3) + 1/2 = -3/2 < \hat{\Xi}(G^{\alpha_1})$.

PROPOSITION 2.4: *Let* $G^\alpha = \langle X | R \rangle$, $X = \{x_1, \ldots, x_n\}$.

(i) *If* $G^{\alpha_1} = \langle X \cup \{x'\} | R, x' = w(X) \rangle$, *where* $w(X) \in \langle X \rangle$, *then* $\hat{\Xi}(G^\alpha) - 1 \leq \hat{\Xi}(G^{\alpha_1}) \leq \hat{\Xi}(G^\alpha)$, *with* $\hat{\Xi}(G^{\alpha_1}) = \hat{\Xi}(G^\alpha)$ *if* $w = 1$.

(ii) *If* $G^{\alpha_2} = \langle X \cup \{x'_1, \ldots, x'_n\} | R, x'_i = x_i, i = 1, \ldots, n \rangle$ *then* $\hat{\Xi}(G^{\alpha_2}) = 2\hat{\Xi}(G^\alpha) - 1/|G|$.

Proof: (i) The Cayley graph of G with respect to G^{α_1} is obtained from the Cayley graph of G with respect to G^α by adding the edges in the direction x' from each vertex v to the vertex vw. Thus we can increase $\beta_1(\mathcal{G}')$ by at most $\beta_0(\mathcal{G}')$. The result then follows from (11). When G is (finite or infinite) amenable then $\hat{\Xi}(G^{\alpha_1}) = \hat{\Xi}(G^\alpha)$ by Proposition 2.3. One can also see it directly from (10) by considering the "thickening" of $\mathcal{G}'$ to $\mathcal{G}''$ by adding to it the outer d-boundary, where $d = l(w)$, and noticing that $|E^{X'}_{out}(\mathcal{G}'')|/\beta_0(\mathcal{G}'') \to 0$, where $X' = X \cup \{x'\}$. When $w = 1$ we can make sure that the edges going-out in directions X are the same as those going-out in directions X' and thus obtain $\hat{\Xi}(G^{\alpha_1}) = \hat{\Xi}(G^\alpha)$.

(ii) When G is finite then $\hat{\Xi}(G^{\alpha_2}) = \hat{\Xi}(G^\alpha) = 1/|G|$ by Proposition 2.3. When G is infinite then since the number of out-going edges is doubled we get that $\hat{\Xi}(G^{\alpha_2}) = 2\hat{\Xi}(G^\alpha)$ as $\hat{\Xi}(G^{\alpha_2}) = -\inf_{\mathcal{G}'} |E^X_{out}(\mathcal{G}')|/\beta_0(\mathcal{G}')$, $\mathcal{G}' \in \mathcal{CF}(\mathcal{G})$. ∎

We notice that by Proposition 2.3 and Proposition 2.4 (ii) we get that $\hat{\Xi}(G^\alpha)$ is independent of the presentation if and only if G is amenable.

When G is finite then ξ has, of course, a maximum, and we have seen that the maximum is achieved only at the whole graph $\mathcal{G}$ of the presentation. We showed also that in general (for finite and infinite groups), for every proper subgraph $\mathcal{H}$ of $\mathcal{G}$ there is a subgraph $\mathcal{H}'$ which properly contains $\mathcal{H}$ and such that $\xi(\mathcal{H}') \geq \xi(\mathcal{H})$. We then ask more: does ξ have a maximum in case the group is infinite? It is quite clear that when G is the free product of a finite group and a free group, with a "natural" presentation so that we have generators of the free part which are not involved in any relator, ξ does have a maximum because no circuit involves the generators of the free factor (for more details on the value of ξ on free products see section 5). Next we will show that in fact this is the only situation where ξ

has a maximum.

Given a presentation $G^\alpha = \langle X | R\rangle$, with $X = \{x_1, \ldots, x_n\}$ and R not empty, let $c(x_i)$ be the length of a shortest relator in which x_i appears (i.e. the shortest (reduced) circuit in the Cayley graph which contains the edge x_i) or 0 if x_i does not appear in any relator. Then let $c(G^\alpha) = \max_i(c(x_i))$.

THEOREM 2.5: *Let $G^\alpha = \langle X | R\rangle$, $|X| = n$. Then one of the following holds.*

Case 1: *$\hat{\Xi}(G^\alpha) = 1 - n + \xi(\mathcal{G}')$ for some $\mathcal{G}' \in \mathcal{F}(\mathcal{G})$. Let $X' \subseteq X$ be the set of labels of the edges not going-out of $\mathcal{G}'$. Then*

(i) *if $X' = X$ then $\mathcal{G}' = \mathcal{G}$ and G is finite;*

(ii) *if $X' = \emptyset$ then G is the free group on X;*

(iii) *if $|X'| = k$, $1 \le k \le n-1$, then $V(\mathcal{G}')$ is the union of left cosets of the finite subgroup $H = Gp(X')$ (which may be trivial) and $G = H * F$ where F is the free group on $X - X'$.*

Case 2: *ξ does not have a maximum on $\mathcal{F}(\mathcal{G})$. Then for every $\mathcal{G}' \in \mathcal{F}(\mathcal{G})$*

$$\hat{\Xi}(G^\alpha) \ge 1 - n + \xi(\mathcal{G}') + \frac{1}{(c(G^\alpha) - 1)\beta_0(\mathcal{G}')}. \tag{23}$$

Proof:

CASE 1: It suffices to show that whenever there is a relator involving an edge going-out of $\mathcal{G}'$ then $\Xi(G^\alpha) > \xi(\mathcal{G}')$. For suppose to the contrary that $\Xi(G^\alpha) = \xi(\mathcal{G}')$ and that $\lambda = v_0, e_1, v_1, e_2, \ldots, e_m, v_0$ is a simple circuit starting at $v_0 \in \partial\mathcal{G}'$ and that $e_1 \in E^X_{out}(\mathcal{G}')$. Note that $m \ge 2$ since $\xi(\mathcal{G}')$ is maximal. Then by Lemma 2.2 we can add translates of $\mathcal{G}''$ along the vertices of λ which are not in $\mathcal{G}'$ so that the resulting subgraph $\mathcal{G}''$ will satisfy $\xi(\mathcal{G}'') \ge \xi(\mathcal{G}')$. Moreover, if $\mathcal{G}''$ does not contain the edge e_1 then by adding this edge to $\mathcal{G}''$ we increase ξ — in contradiction to assumption. To see that this is indeed possible, let $y_k \in X \cup X^{-1}$ be the label of e_k. Then we need to show that for every k for which $v_k \notin V(\mathcal{G}')$, there exists $u_k \in V(\mathcal{G}')$ such that $u_k w_k^{-1} \notin V(\mathcal{G}')$, where $w_k = y_1 \cdots y_k$ (here we look at the vertices as the group elements they represent). But this follows by the assumption that $V(\mathcal{G}')$ is not mapped to itself by the map $g \mapsto g w_k$.

CASE 2: Let $\mathcal{H}'_0 \in \mathcal{F}(\mathcal{G})$ and assume that $\xi(\mathcal{H}'_0) \ge \xi(\mathcal{H}')$ for every $\mathcal{H}' \subseteq \mathcal{H}'_0$, or otherwise we will take such a subgraph $\mathcal{H}' \subseteq \mathcal{H}'_0$ and get a better result. Then, as we have shown above, there exists a subgraph $\mathcal{H}'_1$, which is a union of translates of $\mathcal{H}'_0$, with $\beta_0(\mathcal{H}'_1) \le c(G^\alpha)\beta_0(\mathcal{H}'_0)$ and with

$$\xi(\mathcal{H}'_1) \ge \xi(\mathcal{H}'_0) + \frac{1}{c(G^\alpha)\beta_0(\mathcal{H}'_0)}. \tag{24}$$

The same process can now be carried out with $\mathcal{H}'_1$ and so on, obtaining a sequence $\mathcal{H}'_i$ of subgraphs satisfying

$$\xi(\mathcal{H}'_i) \ge \xi(\mathcal{H}'_{i-1}) + \frac{1}{c(G^\alpha)^i\beta_0(\mathcal{H}'_0)}. \tag{25}$$

Thus, passing to the limit, we get that

$$\hat{\Xi}(G^\alpha) \ge 1 - n + \xi(\mathcal{H}'_0) + \frac{1}{(c(G^\alpha)-1)\beta_0(\mathcal{H}'_0)}. \tag{26}$$ ∎

We remark that in case 2 of the above theorem we get in particular, by taking $\mathcal{G}'$ to be the trivial subgraph, that

$$\hat{\Xi}(G^\alpha) \ge 1 - n + \frac{1}{(c(G^\alpha)-1)} \tag{27}$$

and that there exists an exhausting chain $(\mathcal{H}'_i)$, $\mathcal{H}'_i \in \mathcal{CF}(\mathcal{G})$, such that

$$\limsup_{i\to\infty} \frac{|E^X_{out}(\mathcal{H}'_i)|}{\beta_0(\mathcal{H}'_i)} \le n - 1 - \frac{1}{(c(G^\alpha)-1)}. \tag{28}$$

3. Factor groups and subgroups

When G_2 is a homomorphic image of G, with the presentation $G_2^{\alpha_2}$ induced by the presentation G^α, then the Cayley graph associated with $G_2^{\alpha_2}$ may be regarded as a quotient of the Cayley graph associated with G^α, which implies, as expected, that $\hat{\Xi}(G_2^{\alpha_2}) \ge \hat{\Xi}(G^\alpha)$. In fact we have the following.

THEOREM 3.1: *Let G_1 be a normal subgroup of G and let $G_2 = G/G_1$ with the presentation $G_2^{\alpha_2}$ induced by G^α. Then*

$$\hat{\Xi}(G^\alpha) \le \hat{\Xi}(G_2^{\alpha_2}) - \left(\frac{1}{|G_2|} - \frac{1}{|G|}\right), \tag{29}$$

with equality holding if G_1 is amenable.

Proof: We may exclude the cases where G_1 or G_2 are trivial. When G is finite the result follows by Proposition 2.3 since $\hat{\Xi}(G^\alpha) = 1/|G|$. So assume G is infinite. We will prove first the inequality in (29). Again by Proposition 2.3

the result is clear when G_2 is finite. So we further assume that G_2 is infinite. Let $\mathcal{G}' \in \mathcal{CF}(\mathcal{G})$. We will show there exists a subgraph $\mathcal{G}_2' \in \mathcal{F}(\mathcal{G}_2)$ such that $\beta_1(\mathcal{G}_2')/\beta_0(\mathcal{G}_2') \geq \beta_1(\mathcal{G}')/\beta_0(\mathcal{G}')$. Let p: $\mathcal{G} \to \mathcal{G}_2$ be the covering map from the Cayley graph of G^α onto that of $G_2^{\alpha_2}$. Regarded as a topological space, $\mathcal{G}'$ decomposes into a finite number of subspaces $\mathcal{E}_i'$, where each $\mathcal{E}_i'$ is in a different sheet, that is the map $p|_{\mathcal{E}_i'}$: $\mathcal{E}_i' \to \mathcal{E}_i''$ is bijective and continuous but not necessarily a homeomorphism. Let $\mathcal{G}'' \subseteq \mathcal{G}'$ be a collection of vertices and (open) edges such that the map $p|_{\mathcal{G}''}$ is injective and onto $p(\mathcal{G}')$. In particular, $\beta_0(\mathcal{G}'') = \beta_0(p(\mathcal{G}''))$ and $\beta_1(\mathcal{G}'') = \beta_1(p(\mathcal{G}''))$ (we count here open edges, i.e. not including the endpoints). If $\mathcal{G}'' = \mathcal{G}'$ we are done. Otherwise, we take $\mathcal{G}''' = \mathcal{G}' - \mathcal{G}''$ (which is not necessarily a subgraph) and continue as before. That is, at this step we look at the subgraph $p(\mathcal{G}''')$ and a collection of vertices and edges of $\mathcal{G}'''$ which is mapped injectively onto it. We continue until we cover the whole of $\mathcal{G}'$, and by the finiteness of $\mathcal{G}'$ the process terminates after finitely many steps. There is one thing left to be checked: that $p(\mathcal{G}''')$ and the following projections are subgraphs, i.e. that they are closed: for each open edge the corresponding initial and final vertices are also included. But this follows from the fact that if $v \in V(p(\mathcal{G}'))$, and $e \in E(p(\mathcal{G}'))$ is an edge with v being its endpoint, then there are at least as much vertices in $p^{-1}(v)$ (each in a different sheet) as there are edges in $p^{-1}(e)$. By the way we constructed the sequence of subsets of $\mathcal{G}'$, this property holds throughout the whole process, thus in each step the projected subspace is a subgraph. Since we covered $\mathcal{G}'$ entirely, at least one of these subgraphs $\mathcal{G}_2' \in \mathcal{F}(\mathcal{G}_2)$ satisfies $\beta_1(\mathcal{G}_2')/\beta_0(\mathcal{G}_2') \geq \beta_1(\mathcal{G}')/\beta_0(\mathcal{G}')$. This is true for every $\mathcal{G}' \in \mathcal{CF}(\mathcal{G})$, thus $\hat{\Xi}(G^\alpha) \leq \hat{\Xi}(G_2^{\alpha_2})$ by the second definition in (13).

Suppose now that G_1 is amenable. Then we need to show that $\hat{\Xi}(G^\alpha) = \hat{\Xi}(G_2^{\alpha_2})$ in case G_2 is infinite, or that G is amenable in case G_2 is finite. Let X be the generating set of G^α and let $\mathcal{G}, \mathcal{G}_2$ be the Cayley graphs associated with $G^\alpha, G_2^{\alpha_2}$ respectively. Given $\epsilon > 0$, let $\mathcal{G}_2' \in \mathcal{CF}(\mathcal{G}_2)$ such that

$$\frac{1 - |E^X_{out}(\mathcal{G}_2')|}{\beta_0(\mathcal{G}_2')} > \hat{\Xi}(G_2^{\alpha_2}) - \frac{\epsilon}{3}. \tag{30}$$

Assume also that $\mathcal{G}_2'$ is induced, contains the vertex 1, and that $\mathcal{G}_2' = \mathcal{G}_2$ in case G_2 is finite and otherwise $\beta_0(\mathcal{G}_2') > 3/\epsilon$. Let $\mathcal{T}_2' \in \mathcal{CF}(\mathcal{G}_2)$ be a spanning tree of $\mathcal{G}_2'$. Each vertex of $\mathcal{T}_2'$ is then assigned a specific element of G (of which we make use in (31) below). $\mathcal{T}_2'$ is embedded as a tree $\mathcal{T}' \in \mathcal{CF}(\mathcal{G})$, $\mathcal{T}' \subseteq p^{-1}(\mathcal{T}_2')$ (p the covering map) with the same vertex and edge labels. Then we take $\mathcal{G}' \in \mathcal{CF}(\mathcal{G})$

to be the subgraph induced by T'. Let H_1 be the subgroup of G_1 generated by the set Y consisting of the non-trivial elements

$$(31)\qquad y_{v,x} = vx(p(vx))^{-1} \neq 1, \quad v \in V(T'), \quad x \in X, \quad p(vx) \in V(T').$$

If Y is empty we take H_1 to be the trivial group. Let $\mathcal{H}_1$ be the Cayley graph of H_1 with respect to Y, and let $\mathcal{H}_1' \in \mathcal{CF}(\mathcal{H}_1)$ with

$$(32)\qquad \frac{1 - |E^Y_{out}(\mathcal{H}_1')|}{\beta_0(\mathcal{H}_1')} > -\frac{\epsilon\beta_0(\mathcal{G}_2')}{3}.$$

Such a subgraph exists by the amenability of H_1. Let $\mathcal{G}'' \in \mathcal{F}(\mathcal{G})$ be the subgraph induced by (the disjoint union) $\bigcup_{g\in V(\mathcal{H}_1')} g\mathcal{G}'$ (we look here at $g \in V(\mathcal{H}_1')$ as an element of G by writing each $y \in Y$ with the generators X of G). Let now $e \in E^X_{out}(\mathcal{G}'')$ be an edge labeled with $x \in X$ and starting at $gv \in V(\mathcal{G}'')$, $g \in V(\mathcal{H}_1')$, $v \in V(T')$. Then either $p(e) \in E^X_{out}(\mathcal{G}_2')$, or else $p(e)$ joins $v = p(gv)$ and $u = p(gu)$, for some $u \in V(T')$. Then

$$(33)\qquad gvx = gy_{v,x}p(vx) = gy_{v,x}u.$$

That is, e is the unique edge corresponding to an edge $e' \in E^Y_{out}(\mathcal{H}_1')$ that starts at g and is in direction $y_{v,x}$, and this correspondence is 1-1. Then we have

$$(34)\qquad \begin{aligned} \frac{1 - |E^X_{out}(\mathcal{G}'')|}{\beta_0(\mathcal{G}'')} &= \frac{1 - (\beta_0(\mathcal{H}_1')|E^X_{out}(\mathcal{G}_2')| + |E^Y_{out}(\mathcal{H}_1')|)}{\beta_0(\mathcal{H}_1')\beta_0(\mathcal{G}_2')} \\ &= \frac{1 - |E^X_{out}(\mathcal{G}_2')|}{\beta_0(\mathcal{G}_2')} + \frac{1 - |E^Y_{out}(\mathcal{H}_1')|}{\beta_0(\mathcal{H}_1')\beta_0(\mathcal{G}_2')} - \frac{1}{\beta_0(\mathcal{G}_2')} \\ &> \hat{\Xi}(G_2^{\alpha_2}) - \left(\frac{1}{|G_2|} - \frac{1}{|G|}\right) - \epsilon, \end{aligned}$$

since G is infinite and $|E^X_{out}(\mathcal{G}_2')| = 0$ if G_2 is finite. That is, $\hat{\Xi}(G^\alpha) \geq \hat{\Xi}(G_2^{\alpha_2})$ if G_2 is infinite, and $\hat{\Xi}(G^\alpha) \geq 0$ if G_2 is finite. By the the inequalities in the other directions — these are equalities. ∎

COROLLARY 3.2: *If $G^\alpha = G_1^{\alpha_1} \rtimes G_2^{\alpha_2}$ and G_1 is amenable then*

$$(35)\qquad \hat{\Xi}(G^\alpha) = \hat{\Xi}(G_2^{\alpha_2}) - \left(\frac{1}{|G_2|} - \frac{1}{|G|}\right).$$

Proof: By Proposition 2.4 we may assume that the presentation $G_2^{\alpha_2}$ is induced by the presentation G^α, by adding the generators g_i of $G_1^{\alpha_1}$ and the relations $g_i = 1$ for every i. Then the result follows immediately by Theorem 3.1. ∎

Another corollary to Theorem 3.1 is the known fact that if both G_1 and G_2 are amenable then G is also amenable.

The proof of the first part of Theorem 3.1 implies the theorem of Lubotzky and Weiss (Theorem 3.1 in [10]) that an infinite family of finite quotient groups G_i of a finitely generated amenable group G is a non-expander family. In fact, the theorem of Lubotzky and Weiss says a little more: if G^α is a presentation of G and $\mathcal{G}_i$ are the Cayley graphs of the induced presentations of the quotient groups G_i, then for every $\epsilon > 0$ there exists j such that $\mathcal{G}_j$ is not an ϵ-expander. This means there is a subgraph $\mathcal{G}'$ of $\mathcal{G}_j$ with $\beta_0(\mathcal{G}') \leq |G_j|/2$ and such that the outer boundary of $\mathcal{G}'$ is of size less than $\epsilon\beta_0(\mathcal{G}')$. We may as well look at $|E^X_{out}(\mathcal{G}')|$ instead of the outer boundary. But when G is amenable then for every $\epsilon > 0$ there exists a finite subgraph $\mathcal{G}''$ of the Cayley graph of G such that $|E^X_{out}(\mathcal{G}'')|/\beta_0(\mathcal{G}'') < \epsilon$. When the order of a finite quotient group G_j of G is then large enough (at least twice the cardinality of $\mathcal{G}''$), then, as shown in the proof of Theorem 3.1, the image of $\mathcal{G}''$ in the Cayley graph $\mathcal{G}_j$ of G_j contains a subgraph $\mathcal{G}'$ which satisfies the above inequality (using the equivalence of definitions (10) and (11) of $\hat{\Xi}$). In fact, the proof given in [10], although different from ours, also uses Følner's criterion for amenability, so in this sense there is no great difference. A minor remark: whereas in [10] the result for a quotient group is about being $(2\epsilon)^{1/2}$-invariant, we show it is ϵ-invariant, giving a slightly better bound (this does not affect amenable groups where ϵ can be chosen arbitrary small). (The reader who is interested with this subject of expanding graphs is referred to [11].)

Example 3.3: We have seen in Theorem 3.1 that equality holds when G_1 is amenable. But G_1 may be non-amenable and still $\hat{\Xi}(G^\alpha) = \hat{\Xi}(G_2^{\alpha_2})$. For example, let $G = H * H * K$ where H is a 2-generated finite group and K is free of rank 2. Let K_1 be a normal subgroup of K such that $K/K_1 \simeq H$, let G_1 be the normal closure of K_1 in G, and let $G_2 = G/G_1$. Then $G_2 \simeq H * H * H$ and by Corollary 5.5 (i), $\hat{\Xi}(G) = \hat{\Xi}(G_2)$ although G_1 is non-amenable. This situation does not happen when considering the spectral radii R, R_2 associated with symmetric random walks on G, G_2 respectively, where $R = R_2$ if and only if G_1 is amenable (see [8], Theorem 2).

Definition 3.4: Let G^α be a presentation of G with a generating set X. Let T^α be a Schreier transversal for a subgroup G_1 of G with respect to G^α. Then a **Schreier generating system** Y for G_1 with respect to T^α consists of the non-trivial (in G) elements of the form

$$y_{v,x} = vx(p(vx))^{-1}, \quad v \in T, \quad x \in X, \tag{36}$$

where p is the coset map. Notice that the $y_{v,x}$ are not necessarily distinct elements of G.

PROPOSITION 3.5: *Let G^{α} be a presentation of a group G and let $G_1^{\alpha_1}$ be a presentation by a Schreier generating system of a subgroup G_1 of G of finite index. Then*

$$\hat{\Xi}(G_1^{\alpha_1}) \leq |G:G_1|\hat{\Xi}(G^{\alpha}). \tag{37}$$

Proof: If G is finite then $\hat{\Xi}(G_1^{\alpha_1}) = |G:G_1|\hat{\Xi}(G^{\alpha})$. Assume that G is infinite. Let X be the set, of cardinality n, of generators of G^{α}, and let Y be the Schreier generating system of $G_1^{\alpha_1}$, which is of cardinality $\leq 1+|G:G_1|(n-1)$. Let $\mathcal{G}, \mathcal{G}_1$ be the Cayley graphs associated with $G^{\alpha}, G_1^{\alpha_1}$ respectively. Let $\mathcal{T}' \in \mathcal{CF}(\mathcal{G})$ be the Schreier tree by which Y is defined, and let $\mathcal{G}' \in \mathcal{CF}(\mathcal{G})$ be the subgraph induced by $\mathcal{T}'$. Given $\epsilon > 0$, let $\mathcal{G}_1' \in \mathcal{CF}(\mathcal{G}_1)$ satisfy

$$\frac{1-|E_{out}^{Y}(\mathcal{G}_1')|}{\beta_0(\mathcal{G}_1')} > \hat{\Xi}(G_1^{\alpha_1}) - |G:G_1|\epsilon, \tag{38}$$

and let $\mathcal{G}'' \in \mathcal{F}(\mathcal{G})$ be the subgraph induced by $\bigcup_{g \in V(\mathcal{G}_1')} g\mathcal{G}'$. The edges $E_{out}^{X}(\mathcal{G}'')$ are in 1-1 correspondence with the edges $E_{out}^{Y}(\mathcal{G}_1')$. Then

$$\frac{1-|E_{out}^{X}(\mathcal{G}'')|}{\beta_0(\mathcal{G}'')} = \frac{1-|E_{out}^{Y}(\mathcal{G}_1')|}{|G:G_1|\beta_0(\mathcal{G}_1')} > \frac{\hat{\Xi}(G_1^{\alpha_1})}{|G:G_1|} - \epsilon. \tag{39}$$

Thus we showed that

$$\hat{\Xi}(G_1^{\alpha_1}) \leq |G:G_1|\hat{\Xi}(G^{\alpha}). \quad \blacksquare \tag{40}$$

For example, when G is free then we get an equality in Proposition 3.5.

4. Direct products

THEOREM 4.1: *Let $G_i^{\alpha_i}$ be presentations of non-trivial groups G_i, for $i=1,2$, with disjoint generating sets of cardinalities n_i respectively, and let G^{α} be the induced presentation of $G = G_1 \times G_2$. Then*

$$\hat{\Xi}(G^{\alpha}) = \hat{\Xi}(G_1^{\alpha_1}) + \hat{\Xi}(G_2^{\alpha_2}) - \left(\frac{1}{|G_1|} + \frac{1}{|G_2|} - \frac{1}{|G|}\right). \tag{41}$$

Proof: The claim is true when both G_1 and G_2 are finite, because then

$$\hat{\Xi}(G^\alpha) = \frac{1}{|G|}. \tag{42}$$

When at least one of the groups is infinite we will show first that (41) is an upper bound for $\hat{\Xi}(G^\alpha)$. Suppose that $\mathcal{G}' \in \mathcal{CF}^*(\mathcal{G})$, where $\mathcal{G}$ is the Cayley graph of G. Then $\mathcal{G}'$ is the union of the subgraphs $\mathcal{H}'_1, \mathcal{H}'_2$, where $\mathcal{H}'_i$, $i = 1, 2$, is the subgraph generated by the edges with labels in $X_i \cup X_i^{-1}$. We may also assume that both $\mathcal{H}'_i$ are not empty, because otherwise we can obtain at most $\max(\hat{\Xi}(G_1^{\alpha_1}) - n_2, \hat{\Xi}(G_2^{\alpha_2}) - n_1)$ which is less than or equals the right hand side of (41). Then

$$\mu(\mathcal{G}') - (n_1 + n_2) \le (\mu(\mathcal{H}'_1) - n_1) + \left(\frac{\beta_1(\mathcal{H}'_2)}{\beta_0(\mathcal{H}'_2)} - n_2\right). \tag{43}$$

By (11)

$$\mu(\mathcal{H}'_1) - n_1 \le \hat{\Xi}(G_1^{\alpha_1}), \tag{44}$$

and

$$\frac{\beta_1(\mathcal{H}'_2)}{\beta_0(\mathcal{H}'_2)} - n_2 \le \min(\hat{\Xi}(G_2^{\alpha_2}), 0). \tag{45}$$

Therefore, by symmetry, we get from (43) that

$$\hat{\Xi}(G^\alpha) \le \min(\hat{\Xi}(G_1^{\alpha_1}), \hat{\Xi}(G_2^{\alpha_2}), \hat{\Xi}(G_1^{\alpha_1}) + \hat{\Xi}(G_2^{\alpha_2})). \tag{46}$$

But these bounds are exactly the ones that appear in (41) in case at least one of the groups is infinite.

It remains to show that (41) is really achieved. Given $\epsilon > 0$ then for $i = 1, 2$ let $\mathcal{H}_i \in \mathcal{CF}(\mathcal{G}_i)$, where $\mathcal{G}_i$ is the Cayley graph of $G_i^{\alpha_i}$, satisfying

$$\mu(\mathcal{H}_i) - n_i > \hat{\Xi}(G_i^{\alpha_i}) - \frac{\epsilon}{4}. \tag{47}$$

Suppose also that if G_i is finite then $\mathcal{H}_i = \mathcal{G}_i$ and otherwise $\beta_0(\mathcal{H}_i) > 4/\epsilon$. Let $\mathcal{G}'$ be the subgraph of $\mathcal{G}$ which is the cartesian product $\mathcal{H}_1 \times \mathcal{H}_2$. Then

$$\begin{aligned} \mu(\mathcal{G}') - (n_1 + n_2) &= \frac{\beta_1(\mathcal{H}_1)\beta_0(\mathcal{H}_2) + \beta_1(\mathcal{H}_2)\beta_0(\mathcal{H}_1) + 1}{\beta_0(\mathcal{H}_1)\beta_0(\mathcal{H}_2)} - (n_1 + n_2) \\ &= (\mu(\mathcal{H}_1) - n_1) + (\mu(\mathcal{H}_2) - n_2) - \left(\frac{1}{\beta_0(\mathcal{H}_1)} + \frac{1}{\beta_0(\mathcal{H}_2)} - \frac{1}{\beta_0(\mathcal{H}_1)\beta_0(\mathcal{H}_2)}\right) \qquad (48) \\ &> \hat{\Xi}(G_1^{\alpha_1}) + \hat{\Xi}(G_2^{\alpha_2}) - \left(\frac{1}{|G_1|} + \frac{1}{|G_2|} - \frac{1}{|G|}\right) - \epsilon, \end{aligned}$$

and the proof is complete. ∎

5. Free products

Let us look at what happens with the computation of $\hat{\Xi}(G^\alpha)$ for free products. We recall that the decomposition of a group G into non-trivial freely indecomposable factors is unique, up to isomorphism of the factors, as follows by the Kurosh Subgroup Theorem. Such a decomposition contains finitely many factors when G is of finite rank, and in fact, by the corollary to the Grushko–Neumann Theorem, if $G = G_1 * G_2$ then $\mathrm{rank}(G) = \mathrm{rank}(G_1) + \mathrm{rank}(G_2)$ (see [12], p. 178). The following expression plays an important role in computing the value of $\hat{\Xi}(G^\alpha)$.

Definition 5.1: Let G^α be an n-generated presentation of a non-trivial group G. If $\mathcal{G}' \in \mathcal{F}^*(\mathcal{G})$ let

$$\psi(\mathcal{G}') = \frac{\beta_2(\mathcal{G}')}{\beta_0(\mathcal{G}') - 1}. \tag{49}$$

Then we define

$$\Psi(G^\alpha) = \sup_{\mathcal{G}' \in \mathcal{F}^*(\mathcal{G})} \psi(\mathcal{G}') \tag{50}$$

and

$$\hat{\Psi}(G^\alpha) = 1 - n + \Psi(G^\alpha). \tag{51}$$

In contrast to the situation with ξ, it may happen that for some finite subgraph $\mathcal{G}'$ of an infinite graph $\mathcal{G}$ we have $\psi(\mathcal{G}') > \psi(\mathcal{G}'')$ for any other non-trivial finite subgraph $\mathcal{G}''$ (e.g. when G is the free product of cyclic groups of orders 2 and 3 and ψ achieves a maximum on a subgraph of size 2). We call a presentation **reduced** if none of its generators equals the identity element in the group. Since removing such "redundant generators" does not change the value of $\hat{\Xi}(G^\alpha)$, no loss of generality is caused when assuming (as we do in Theorem 5.3) that the presentations are reduced. We say that a presentation $G^\alpha = \langle X \mid R \rangle$ is **minimal** if for every proper subset X' of X, $Gp(X') \neq G$. (Here $Gp(X')$ is the subgroup of G generated by X'.) Note that when G is a finite non-trivial group which has an n-generated minimal presentation then $|G| \geq 2^n$ (by induction on n).

THEOREM 5.2: *Let $G^\alpha = \langle X \mid R \rangle$ be an n-generated minimal presentation of a non-trivial group G. Then $1 - n \leq \hat{\Psi}(G^\alpha) \leq 1$. Moreover,*

- (i) *if G is finite then $\hat{\Psi}(G^\alpha) = n/(|G| - 1)$, with $\hat{\Psi}(G^\alpha) = 1$ if and only if $|G| = 2$;*
- (ii) *if G is infinite then $\hat{\Psi}(G^\alpha) \leq 0$, with $\hat{\Psi}(G^\alpha) = 0$ if and only if G is amenable or G^α is 2-generated and one of the generators is of order 2. $\hat{\Psi}(G^\alpha) = 1 - n$*

if and only if G is free of rank $n \geq 2$.

Let $H = Gp(Y)$, $Y \subseteq X$, satisfying $|H| = \min\{|Gp(X')|\}$, where $X' \subseteq X$ and $|X'| = \max\{|X''|: X'' \subseteq X, |Gp(X'')| < \infty\}$. Then

$$\Psi(G^\alpha) = \max(\Xi(G^\alpha), \Psi(H^\alpha)), \tag{52}$$

where $\Psi(H^\alpha)$ is calculated as in (i).

Proof: Let $\mathcal{G}$ be the Cayley graph of G^α.

$$\begin{aligned}\hat{\Psi}(G^\alpha) &= 1 - n + \sup_{\mathcal{G}' \in \mathcal{CF}^*(\mathcal{G})} \frac{1 + (n-1)\beta_0(\mathcal{G}') - |E^X_{out}(\mathcal{G}')|}{\beta_0(\mathcal{G}') - 1} \\ &= \sup_{\mathcal{G}'} \frac{n - |E^X_{out}(\mathcal{G}')|}{\beta_0(\mathcal{G}') - 1}.\end{aligned} \tag{53}$$

Therefore, $\hat{\Psi}(G^\alpha) \leq 1$ if and only if

$$|E^X_{out}(\mathcal{G}')| \geq n + 1 - \beta_0(\mathcal{G}') \tag{54}$$

for every $\mathcal{G}' \in \mathcal{CF}^*(\mathcal{G})$. Assume that $X = \{x_1, \ldots, x_n\}$, and $X' = \{x_1, \ldots, x_k\}$, $0 \leq k \leq n$, is the set of the labels of the edges **not** going-out of $\mathcal{G}'$. If $k = 0$ then $|E^X_{out}(\mathcal{G}')| \geq n$ and $\hat{\Psi}(G^\alpha) \leq 0$. Otherwise, $\mathcal{G}'$ is the union of a finite number of left cosets of $Gp(X')$. Since the presentation is minimal we have

$$\beta_0(\mathcal{G}') \geq |Gp(X')| \geq 2^k. \tag{55}$$

Hence

$$|E^X_{out}(\mathcal{G}')| \geq n - k \geq n - (2^k - 1) \geq n + 1 - \beta_0(\mathcal{G}'). \tag{56}$$

Suppose that G is finite. We will show that ψ achieves its maximum on $\mathcal{G}$. Let $\mathcal{G}' \in \mathcal{CF}^*(\mathcal{G})$ satisfy $\psi(\mathcal{G}') \geq \psi(\mathcal{H})$ for every $\mathcal{H} \in \mathcal{CF}^*(\mathcal{G})$ contained in $\mathcal{G}'$. Let $\mathcal{G}'' \neq \mathcal{G}'$ be a left translate of $\mathcal{G}'$ such that $V(\mathcal{G}' \cap \mathcal{G}'')$ is not empty, and let $\mathcal{H}' = \mathcal{G}' \cup \mathcal{G}''$. Then

$$\psi(\mathcal{H}') \geq \frac{2\beta_2(\mathcal{G}') - \beta_2(\mathcal{G}' \cap \mathcal{G}'')}{2\beta_0(\mathcal{G}') - \beta_0(\mathcal{G}' \cap \mathcal{G}'') - 1} \geq \psi(\mathcal{G}'), \tag{57}$$

where the right inequality comes from

$$\frac{a}{b-1} \geq \frac{c}{d-1} \iff \frac{2a - c}{2b - d - 1} \geq \frac{a}{b-1}, \tag{58}$$

with all denominators positive (the case where $\mathcal{G}'$ meets $\mathcal{G}''$ in a single vertex leads to an equality in (57)).

It is left to examine the case where for every translate $\mathcal{G}''$ of $\mathcal{G}'$, $\mathcal{G}'' \neq \mathcal{G}'$, $V(\mathcal{G}' \cap \mathcal{G}'')$ is empty. This means that no edge going-out of $\mathcal{G}'$ has the same label as that of an edge of $\mathcal{G}'$. Thus $\mathcal{G}'$ is isomorphic to the Cayley graph of $H = Gp(X')$, $|X'| = k$, and by (53)

$$\psi(\mathcal{G}') = k - 1 + \frac{k}{|H| - 1} \leq k. \tag{59}$$

If H is then a proper subgroup of G and $x_j \notin X'$ then the subgraph $\mathcal{H}'$ which is isomorphic to the Cayley graph of $H' = Gp(X' \cup x_j)$ satisfies

$$\psi(\mathcal{H}') = k + \frac{k+1}{|H'| - 1} > k \geq \psi(\mathcal{G}'). \tag{60}$$

We have shown that $\psi(\mathcal{G}') \leq \psi(\mathcal{G})$ for every subgraph $\mathcal{G}' \in \mathcal{CF}^*(\mathcal{G})$. Hence

$$\hat{\Psi}(G^\alpha) = 1 - n + \psi(\mathcal{G}) = \frac{n}{|G| - 1}. \tag{61}$$

Since by the minimality of the presentation $|G| \geq 2^n$ then $\hat{\Psi}(G^\alpha) = 1$ if and only if G is of order 2.

Suppose now that G is infinite. If $\psi(\mathcal{G}')$ does not have a maximum on $\mathcal{CF}^*(\mathcal{G})$ then $\hat{\Psi}(G^\alpha) = \hat{\Xi}(G^\alpha)$ since $|\psi(\mathcal{G}') - \xi(\mathcal{G}')| \to 0$ as $\beta_0(\mathcal{G}') \to \infty$. The same is true when $\psi(\mathcal{G}')$ does have a maximum but there is no bound to the size of $\mathcal{G}'$ on which ψ achieves its maximum, e.g. when $G = H * H$ and H is finite. In fact, by Corollary 5.5 (vii), when $G = H * H$ then $\Psi(G^\alpha) = \Psi(H^\alpha)$, and if $\mathcal{H}$ is the Cayley graph of H, embedded in $\mathcal{G}$, then when we adjoin m translates of $\mathcal{H}$ to form $\mathcal{K} \in \mathcal{CF}^*(\mathcal{G})$, each translate intersecting the previous one in a single vertex, then

$$\psi(\mathcal{K}) = \frac{m\beta_2(\mathcal{H})}{m\beta_0(\mathcal{H}) - (m-1) - 1} = \psi(\mathcal{H}). \tag{62}$$

When none of the above occurs then ψ achieves its maximum on some $\mathcal{G}' \in \mathcal{CF}^*(\mathcal{G})$, which is isomorphic to the Cayley graph of $H = Gp(X')$, $|X'| = k < n$. Then by (59)

$$\begin{aligned} \hat{\Psi}(G^\alpha) = 1 - n + \psi(\mathcal{G}') &= 1 - n + \left(k - 1 + \frac{k}{|H| - 1}\right) \\ &= k - n + \frac{k}{|H| - 1} \leq 0. \end{aligned} \tag{63}$$

In fact, we see that when G is non-amenable then $\hat{\Psi}(G^\alpha) = 0$ if and only if G^α is 2-generated and one of the generators is of order 2. When G is infinite amenable then $\hat{\Psi}(G^\alpha) = 0$.

We conclude that for both finite and infinite groups G, if $H = Gp(Y)$, $Y \subseteq X$, satisfies $|H| = \min\{|Gp(X')|\}$, where $X' \subseteq X$ and $|X'| = \max\{|X''|: X'' \subseteq X, |Gp(X'')| < \infty\}$, then

$$\Psi(G^\alpha) = \max(\Xi(G^\alpha), \Psi(H^\alpha)), \tag{64}$$

where $\Psi(H^\alpha) = m - 1 + m/(|H| - 1)$, with m being the number of generators of H^α.

Finally, it is clear that $\hat{\Xi}(G^\alpha) = 1 - n$ if and only if G is free of rank $n \geq 2$. ∎

When the presentation is not minimal the assertions of Theorem 5.2 do not hold. For example, if $G^\alpha = \langle x_1, \dots, x_n | \, x_1 = x_2 = \cdots = x_n, \; x_1^2 = 1\rangle$ and $n \geq 2$ then $\Psi(G^\alpha) = 2n - 1 > n$.

THEOREM 5.3: *For each i, $1 \leq i \leq r$, $r \geq 2$, let $G_i^{\alpha_i} = \langle X_i | \, R_i\rangle$ be a reduced n_i-generated presentation of a non-trivial group G_i whose Cayley graph is $\mathcal{G}_i$. Let $G^\alpha = \langle \bigcup_{i=1}^r X_i | \bigcup_{i=1}^r R_i\rangle$ be the induced $n = \sum_{i=1}^r n_i$-generated presentation of $G = G_1 * G_2 * \cdots * G_r$. Assume also, without loss of generality, that $\Psi(G_1^{\alpha_1}) \geq \Psi(G_2^{\alpha_2}) \geq \cdots \geq \Psi(G_r^{\alpha_r})$, and let $G_{1,2}^\alpha$ be the induced presentation of $G_1 * G_2$.*

(i) *If $\Psi(G_1^{\alpha_1}) = \Xi(G_1^{\alpha_1})$ then $\hat{\Xi}(G^\alpha) = 1 - n + \Xi(G_1^{\alpha_1})$.*

(ii) *If $\Psi(G_1^{\alpha_1}) = \Psi(H_1^{\alpha_1}) > \Xi(G_1^{\alpha_1})$, where $H_1 < G_1$ is a finite subgroup generated by $Y_1 \subseteq X_1$ as in Theorem 5.2, then*

$$\hat{\Xi}(G^\alpha) = 1 - n + \Xi(G_{1,2}^\alpha) = 1 - n + \max\left(\Xi(G_1^{\alpha_1}), \Xi(H_1^{\alpha_1}) + \frac{\Psi(G_2^{\alpha_2})}{|H_1|}\right). \tag{65}$$

Proof: Let $\mathcal{G}$ be the Cayley graph corresponding to G^α. Given $\epsilon > 0$ let $\mathcal{G}' \in \mathcal{CF}^*(\mathcal{G})$ satisfying

$$\xi(\mathcal{G}') > \Xi(G^\alpha) - \epsilon. \tag{66}$$

$\mathcal{G}'$ has the form $\mathcal{G}' = \bigcup_i \mathcal{H}_i$, where each $\mathcal{H}_i$ is the disjoint union of $k_i \geq 0$ subgraphs $\mathcal{H}_{i,j} \in \mathcal{CF}^*(\mathcal{G}_i)$, $j = 1, \dots, k_i$, $\mathcal{G}_i$ being the Cayley graph of $G_i^{\alpha_i}$. We say that such a subgraph $\mathcal{H}_{i,j}$ is of type i. Starting from some $\mathcal{H}_{i_0,j_0}$, $\mathcal{G}'$ can be constructed inductively, forming a tree-like structure, by adding at each stage one of the subgraphs $\mathcal{H}_{i,j}$, which meets the subgraph constructed up to that stage at a single vertex, since there are no other simple circuits except the ones in the subgraphs $\mathcal{H}_{i,j}$ (this is where ψ comes into the picture). This implies that

$$\xi(\mathcal{G}') = \frac{\sum_{i=1}^r \beta_2(\mathcal{H}_i)}{\sum_{i=1}^r \beta_0(\mathcal{H}_i) + 1 - \sum_{i=1}^r k_i} = \frac{\sum_{i=1}^r \sum_{j=1}^{k_i} \beta_2(\mathcal{H}_{i,j})}{1 + \sum_{i=1}^r \sum_{j=1}^{k_i} (\beta_0(\mathcal{H}_{i,j}) - 1)}. \tag{67}$$

By the form of (67) we see that an upper bound for $\Xi(G^\alpha)$ is $\Psi(G_1^{\alpha_1})$, and by using only subgraphs of type 1 and 2 we get a lower bound $\Xi(G^\alpha) \geq \Psi(G_2^{\alpha_2})$. In case $\Psi(G_1^{\alpha_1}) = \Psi(G_2^{\alpha_2})$ then $\Xi(G^\alpha) = \Psi(G_1^{\alpha_1})$. We may then assume that $G_1^{\alpha_1}$ is involved in $\mathcal{G}'$ when ϵ is small enough. We may also assume that except from $G_1^{\alpha_1}$, $\mathcal{G}'$ involves edges from other $G_i^{\alpha_i}$ (otherwise we have a connected subgraph $\mathcal{H}$ of $\mathcal{G}_1$ and we can take two copies of it joined by an edge from some X_i, $i \neq 1$, so that $\xi(\mathcal{H})$ is not changed). We look at the decomposition of $\mathcal{G}'$ into the subgraphs $\mathcal{H}_{i,j}$ as above. Then we reconstruct $\xi(\mathcal{G}')$ in the following way. We start from a subgraph $\mathcal{H}_{1,j_0}$ of type 1. Then we add the subgraph of $\mathcal{G}'$ consisting of some $\mathcal{H}_{i_1,j_1}$, of type $i_1 \neq 1$ and all the new subgraphs (not including the one we started with) of type 1 joined to it (and there may be none of them). We continue in an inductive way, so that at the m-th stage we add some new $\mathcal{H}_{i_m,j_m}$, of type $i_m \neq 1$, which is joined to the part constructed up to that stage, and all the new subgraphs of type 1 joined to $\mathcal{H}_{i_m,j_m}$. We finish after we cover the whole of $\mathcal{G}'$. We show now that there exists a subgraph $\mathcal{G}''$ of $\mathcal{G}$, decomposed into copies of only two subgraphs $\mathcal{H} \subseteq \mathcal{G}_1$ and $\mathcal{H}' \subseteq \mathcal{G}_2$, such that $\xi(\mathcal{G}'') \geq \xi(\mathcal{G}')$. First we notice that if for some $\mathcal{H}_{i_m,j_m}$, $i_m \neq 1$, the number of subgraphs of type 1 joined to it is less than $\beta_0(\mathcal{H}_{i_m,j_m})$ then there are subgraphs of type 1 that we can add to it so that ξ does not decrease since $\Psi(G_1^{\alpha_1}) \geq \Psi(G_i^{\alpha_i})$ for every i. So let us suppose that indeed each such $\mathcal{H}_{i_m,j_m}$ is joined to $\beta_0(\mathcal{H}_{i_m,j_m})$ subgraphs of type 1. Let $\mathcal{H}_{1,j_k}$, $k = 1, \ldots, p_m = \beta_0(\mathcal{H}_{i_m,j_m}) - 1$, be the new subgraphs of type 1 added at the m-th stage in the reconstruction of $\mathcal{G}'$. That is, at that stage β_2 is increased by

(68)
$$a_m = \beta_2(\mathcal{H}_{i_m,j_m}) + \sum_{k=1}^{p_m} \beta_2(\mathcal{H}_{1,j_k})$$

and β_0 is increased by

(69)
$$b_m = \sum_{k=1}^{p_m} \beta_0(\mathcal{H}_{1,j_k}).$$

Since $\Psi(G_2^{\alpha_2}) \geq \Psi(G_i^{\alpha_i})$ for every $i > 2$, then there exist finite connected subgraphs $\mathcal{H}'_m$ of type 2 and $\mathcal{H}''_m$ of type 1, such that

$$(70)\qquad \xi(\mathcal{H}''_m)+\frac{\psi(\mathcal{H}'_m)}{\beta_0(\mathcal{H}''_m)}=\frac{(\beta_0(\mathcal{H}'_m)-1)\beta_2(\mathcal{H}''_m)+\beta_2(\mathcal{H}'_m)}{(\beta_0(\mathcal{H}'_m)-1)\beta_0(\mathcal{H}''_m)} \geq \frac{(\beta_0(\mathcal{H}_{i_m,j_m})-1)\beta_2(\mathcal{H}_{1,j_k})+\beta_2(\mathcal{H}_{i_m,j_m})}{(\beta_0(\mathcal{H}_{i_m,j_m})-1)\beta_0(\mathcal{H}_{1,j_k})}$$

for every $1\leq m\leq t=\sum_{i=2}^{r}k_i$ and $1\leq k\leq p_m$ (first choose $\mathcal{H}'_m$ such that $\psi(\mathcal{H}'_m)\geq\psi(H_{i_m,j_m})$ for every m, and then an appropriate $\mathcal{H}''_m$). Hence it follows that

$$(71)\qquad \xi(\mathcal{H}''_m)+\frac{\psi(\mathcal{H}'_m)}{\beta_0(\mathcal{H}''_m)}\geq\frac{a_m}{b_m}$$

(as seen after clearing denominators). Let $\mathcal{H}\in\mathcal{CF}(\mathcal{G}_1),\mathcal{H}'\in\mathcal{CF}(\mathcal{G}_2)$ such that

$$(72)\qquad \xi(\mathcal{H})+\frac{\psi(\mathcal{H}')}{\beta_0(\mathcal{H})}\geq\xi(\mathcal{H}''_m)+\frac{\psi(\mathcal{H}'_m)}{\beta_0(\mathcal{H}''_m)}$$

for each $m=1,\ldots,t$. By the last two inequalities the subgraph $\mathcal{G}''$ constructed by starting with $\mathcal{H}_{1,j_0}$ and adjoining t times the subgraph consisting of a copy of $\mathcal{H}'$ and $(\beta_0(\mathcal{H}')-1)$ copies of $\mathcal{H}$ satisfies

$$(73)\qquad \xi(\mathcal{G}'')\geq\xi(\mathcal{G}').$$

Since ϵ was chosen arbitrarily, we get by the form of $\mathcal{G}''$ that

$$(74)\qquad \begin{aligned}\hat{\Xi}(G^\alpha)&=1-n+\sup_{\mathcal{H}'_1\in\mathcal{CF}(\mathcal{G}_1),\mathcal{H}'_2\in\mathcal{CF}(\mathcal{G}_2)}\left(\xi(\mathcal{H}'_1)+\frac{\psi(\mathcal{H}'_2)}{\beta_0(\mathcal{H}'_1)}\right)\\ &=1-n+\sup_{\mathcal{H}'_1\in\mathcal{CF}(\mathcal{G}_1)}\left(\xi(\mathcal{H}'_1)+\frac{\Psi(G_2^{\alpha_2})}{\beta_0(\mathcal{H}'_1)}\right)\\ &=1-n+\Xi(G^\alpha_{1,2}).\end{aligned}$$

Let us define $\zeta(\mathcal{H}'_1)$ on $\mathcal{CF}(\mathcal{G}_1)$ by

$$(75)\qquad \zeta(\mathcal{H}'_1)=\xi(\mathcal{H}'_1)+\frac{\Psi(G_2^{\alpha_2})}{\beta_0(\mathcal{H}'_1)}.$$

Thus we need to find

$$(76)\qquad \Xi(G^\alpha)=\sup_{\mathcal{H}'_1\in\mathcal{CF}(\mathcal{G}_1)}\zeta(\mathcal{H}'_1).$$

If $\Psi(G_1^{\alpha_1})=\Psi(G_2^{\alpha_2})$ then as we have seen $\Xi(G^\alpha)=\Psi(G_1^{\alpha_1})=\zeta(1)$, where 1 is the trivial subgraph. Also, when $\Xi(G_1^{\alpha_1})=\Psi(G_1^{\alpha_1})$ then $\Xi(G^\alpha)=\Xi(G_1^{\alpha_1})$ because $\Psi(G^\alpha)=\Psi(G_1^{\alpha_1})$ by (67) and $\Xi(G^\alpha)\geq\Xi(G_1^{\alpha_1})$ by the embedding of

$\mathcal{G}_1$ in $\mathcal{G}$. When $\Xi(G_1^{\alpha_1}) < \Psi(G_1^{\alpha_1})$ but an arbitrarily large subgraph $\mathcal{H}_1'$ can be chosen without decreasing $\zeta(\mathcal{H}_1')$ then we get that $\Xi(G^\alpha) = \Xi(G_1^{\alpha_1})$ as the second summand in the expression defining $\zeta(\mathcal{H}_1')$ tends to zero when $\beta_0(\mathcal{H}_1') \to \infty$. It remains to check the case where ζ achieves its maximum on a finite number of members of $\mathcal{CF}(\mathcal{G}_1)$. Let $\mathcal{H}_1'$ be maximal (with respect to the number of vertices) among these subgraphs. We will show that $\mathcal{H}_1'$ is isomorphic to the Cayley graph of a finite subgroup H_1 of G_1 of the form described in Theorem 5.2 (ii) on which ψ achieves its maximum. For we are given that $\mathcal{H}_1'$ satisfies $\zeta(\mathcal{H}_1') \geq \zeta(\mathcal{H}')$ for every $\mathcal{H}' \in \mathcal{CF}(\mathcal{G})$ which is contained in $\mathcal{H}_1'$. Suppose there exists $\mathcal{H}_1'' \neq \mathcal{H}_1'$ a left translate of $\mathcal{H}_1'$ such that $\mathcal{H}_1' \cap \mathcal{H}'' \neq \emptyset$. Let $\mathcal{H}_1 = \mathcal{H}_1' \cup \mathcal{H}_1''$. Then

$$\zeta(\mathcal{H}_1) \geq \frac{2\beta_2(\mathcal{H}_1') - \beta_2(\mathcal{H}_1' \cap \mathcal{H}_1'') + \Psi(G_2^{\alpha_2})}{2\beta_0(\mathcal{H}_1') - \beta_0(\mathcal{H}_1' \cap \mathcal{H}_1'')} \geq \zeta(\mathcal{H}_1'), \tag{77}$$

where the right inequality comes from

$$\frac{a+\Psi}{b} \geq \frac{c+\Psi}{d} \iff \frac{2a-c+\Psi}{2b-d} \geq \frac{a+\Psi}{b}, \tag{78}$$

with all denominators positive. But this contradicts the maximality of $\mathcal{H}_1'$. Thus the set of labels of the edges of $\mathcal{H}_1'$ is disjoint from the set of labels of its outer edges. This means that $\mathcal{H}_1'$ is isomorphic to the Cayley graph of a subgroup of G_1. We need to show that ψ too achieves its maximum on $\mathcal{H}_1'$. Recall that we are in the case where $\Psi(G_1^{\alpha_1}) > \Xi(G_1^{\alpha_1})$, and so ψ achieves its maximum on some subgraph $\mathcal{H}_1$ which is isomorphic to the Cayley graph of a finite subgroup $H_1 < G_1$. Let $\mathcal{K}_1 \in \mathcal{CF}^*(\mathcal{G}_1)$ be isomorphic to the Cayley graph of a finite subgroup $K_1 < G_1$. Thus $\Psi(H_1^{\alpha_1}) \geq \Psi(K_1^{\alpha_1})$. If $|H_1| \leq |K_1|$ then since for finite groups

$$\Psi(H_1^{\alpha_1}) \geq \Psi(K_1^{\alpha_1}) \iff \Xi(H_1^{\alpha_1}) \geq \Xi(K_1^{\alpha_1}), \tag{79}$$

we get that

$$\zeta(\mathcal{H}_1) = \Xi(H_1^{\alpha_1}) + \frac{\Psi(G_2^{\alpha_2})}{|H_1|} \geq \Xi(K_1^{\alpha_1}) + \frac{\Psi(G_2^{\alpha_2})}{|K_1|} = \zeta(\mathcal{K}_1). \tag{80}$$

So assume $|H_1| > |K_1|$. Then $\psi(\mathcal{H}_1) \geq \psi(\mathcal{K}_1)$ is equivalent to

$$\xi(\mathcal{H}_1) + \frac{\psi(\mathcal{H}_1)}{\beta_0(\mathcal{H}_1)} \geq \xi(\mathcal{K}_1) + \frac{\psi(\mathcal{H}_1)}{\beta_0(\mathcal{K}_1)}. \tag{81}$$

Since $\psi(\mathcal{H}_1) \geq \Psi(G_2^{\alpha_2})$ we have

$$\xi(\mathcal{H}_1) - \xi(\mathcal{K}_1) \geq \psi(\mathcal{H}_1)\left(\frac{1}{\beta_0(\mathcal{K}_1)} - \frac{1}{\beta_0(\mathcal{H}_1)}\right) \tag{82}$$

$$\geq \Psi(G_2^{\alpha_2})\left(\frac{1}{\beta_0(\mathcal{K}_1)} - \frac{1}{\beta_0(\mathcal{H}_1)}\right).$$

That is

$$\zeta(\mathcal{H}_1) = \xi(\mathcal{H}_1) + \frac{\Psi(G_2^{\alpha_2})}{\beta_0(\mathcal{H}_1)} \geq \xi(\mathcal{K}_1) + \frac{\Psi(G_2^{\alpha_2})}{\beta_0(\mathcal{K}_1)} = \zeta(\mathcal{K}_1). \tag{83}$$

We conclude that if H_1 is a finite subgroup of G_1 (G_1 may be finite or infinite) generated by some set $Y_1 \subseteq X_1$ satisfying

$$\begin{aligned} |H_1| = \min\{|Gp(X_1')|: X_1' \subseteq X_1, \\ |X_1'| = \max\{|X_1''|: X_1'' \subseteq X_1, |Gp(X_1'')| < \infty\}\} \end{aligned} \tag{84}$$

then

$$\hat{\Xi}(G^\alpha) = 1 - n + \max\left(\Xi(G_1^{\alpha_1}), \Xi(H_1^{\alpha_1}) + \frac{\Psi(G_2^{\alpha_2})}{|H_1|}\right). \tag{85}$$

If, on the other hand, $Gp(X_1')$ is infinite for every $X_1' \subseteq X_1$ then $\hat{\Xi}(G^\alpha) = 1 - n + \Xi(G_1^{\alpha_1})$. ∎

Example 5.4: Let $G = G_1 * G_2$, where $G_1 = C_2 * C_3$ (C_i the cyclic group of order i) and $G_2 = C_4$, with all cyclic factors single-generated. Then $\Psi(G_1) = \Psi(C_2) = 1$, $\Psi(G_2) = 1/3$, $\Xi(G_1) = \Xi(C_2) + \Psi(C_3)/|C_2| = 1/2 + 1/(2\cdot 2) = 3/4$, and $\Xi(G) = \Xi(G_1) = 3/4 > 2/3 = 1/2 + 1/(3\cdot 2) = \Xi(C_2) + \Psi(G_2)/|C_2|$.

On the other hand, suppose that $G_1 = C_2 * C_4$, $G_2 = C_3$ and $G = G_1 * G_2$. Then $\Psi(G_1) = 1$, $\Psi(G_2) = 1/2$, $\Xi(G_1) = 1/2 + 1/(3\cdot 2) = 2/3$ and $\Xi(G) = 3/4 = 1/2 + 1/(2\cdot 2) = 1/\Xi(C_2) + \Psi(G_2)/|C_2| > \Xi(G_1)$. We see that when $\Psi(G_1) > \Xi(G_1)$ then either of the possibilities for $\Xi(G)$ that are stated in Theorem 5.3 can occur.

In the following corollary the results either follow immediately from Theorem 5.3 or are already stated within the proof of Theorem 5.3 or follow from the arguments there. Therefore only a partial proof is given. We further assume that the presentations are reduced.

COROLLARY 5.5: *Let $G = G_1 * G_2 * \cdots * G_r$ with the assumptions of Theorem 5.3. Then the following claims hold.*

(i) $\Psi(G_2^{\alpha_2}) \leq \Xi(G^\alpha) \leq \Psi(G_1^{\alpha_1})$.

(ii) $\Xi(G^\alpha) \geq \max_i\{\Xi(G_i^{\alpha_i})\}$.
$\Xi(G^\alpha) = \Xi(G_1^{\alpha_1})$ *if*
  (a) $\Xi(G_1^{\alpha_1}) = \Psi(G_1^{\alpha_1})$; *or*
  (b) $Gp(X_1')$ *is infinite for every* $X_1' \subseteq X_1$, *in particular if* G_1 *is torsion-free; or*

(c) $\xi(\mathcal{H})$ *does not have a maximum on* $\mathcal{F}(\mathcal{G}_1)$ *and* $(c(G_1^{\alpha_1})-1)\Psi(G_2^{\alpha_2}) \leq 1$; *or*

(d) $\Psi(K_1^{\alpha_1}) \geq \Psi(G_2^{\alpha_2})$, *where* $K_1 = Gp(Z_1)$, $Z_1 = X_1 - Y_1$ *is non-empty and* $Y_1 \subseteq X_1$ *generates a finite subgroup* H_1 *for which* $\Psi(G_1^{\alpha_1}) = \Psi(H_1^{\alpha_1}) > \Xi(G_1^{\alpha_1})$.

(iii) $\hat{\Xi}(G^\alpha) \leq 1 - r + \sum_{i=1}^r \hat{\Xi}(G_i^{\alpha_i})$, *with equality holding if and only if either* G_2 *is free or* $G_1 \simeq G_2 \simeq C_2$ *and* G_3 *(if exists) is free.*

(iv) $\hat{\Xi}(G^\alpha) = n_1 - n$ *if* G_1 *is infinite amenable.*

(v) $\Xi(G^\alpha) = \Xi(G_1^{\alpha_1}) + \frac{\Psi(G_2^{\alpha_2})}{|G_1|} = n_1 - 1 + \frac{\Psi(G_2^{\alpha_2})+1}{|G_1|}$ *if* G_1 *is finite.*
$\Xi(G^\alpha) = n_1 - 1 + \frac{n_2|G_2|}{|G_1|(|G_2|-1)}$ *if* G_1 *and* G_2 *are finite and the presentation of* G_2 *is minimal.*

(vi) $\hat{\Xi}(G^\alpha) = 1 - r + \frac{1}{|G_1|(1-\frac{1}{|G_2|})}$ *if all the factors are (finite or infinite) cyclic and single-generated.*

(vii) $\Psi(G^\alpha) = \Psi(G_1^{\alpha_1})$.

Proof: (ii)(c) For every $\epsilon > 0$ there exists $\mathcal{H} \in \mathcal{CF}(\mathcal{G}_1)$ such that

$$\Xi(G^\alpha) \leq \zeta(\mathcal{H}) + \epsilon = \xi(\mathcal{H}) + \frac{\Psi(G_2^{\alpha_2})}{\beta_0(\mathcal{H})} + \epsilon. \tag{86}$$

When $\xi(\mathcal{H})$ does not have a maximum on $\mathcal{F}(\mathcal{G}_1)$ (which is the case "in general") then by Theorem 2.5

$$\Xi(G_1^{\alpha_1}) \geq \xi(\mathcal{H}) + \frac{1}{(c(G_1^{\alpha_1})-1)\beta_0(\mathcal{H})}. \tag{87}$$

If, in addition, $(c(G_1^{\alpha_1})-1)\Psi(G_2^{\alpha_2}) \leq 1$ then we get from the two inequalities that

$$\Xi(G^\alpha) \leq \Xi(G_1^{\alpha_1})), \tag{88}$$

and by the inequality in the other direction this is an equality.

(ii)(d) $Gp(Y_1 \cup Z_1) < G_1$ is a quotient of $L_1 = H_1 * K_1$ and therefore, by Theorem 3.1 $\Xi(G_1^{\alpha_1}) \geq \Xi(L_1^{\alpha_1}) \geq \Xi(H_1^{\alpha_1}) + \frac{\Psi(L_1^{\alpha_1})}{|H_1|} \geq \Xi(H_1^{\alpha_1}) + \frac{\Psi(G_2^{\alpha_2})}{|H_1|}$. Thus, by Theorem 5.3, $\Xi(G^\alpha) = \Xi(G_1^{\alpha_1})$.

(iii) $\hat{\Xi}(G^\alpha) \leq 1 - r + \sum_{i=1}^r \hat{\Xi}(G_i^{\alpha_i})$ since by (67)

$$\xi(\mathcal{G}') = \sum_{i=1}^r \frac{\beta_2(\mathcal{H}_i)}{\beta_0(\mathcal{G}')} \leq \sum_{i=1}^r \xi(\mathcal{H}_i). \tag{89}$$

A necessary condition for equality in (iii) is that for every $\epsilon > 0$ there exists $\mathcal{G}'$ such that for **every** non-free factor G_i, $\beta_0(\mathcal{H}_i)/\beta_0(\mathcal{G}') > 1 - \epsilon$. But this is possible if and only if there is either only one non-free factor, or there are two such factors and each is isomorphic to C_2, the group of order 2.

(iv) If G_1 is infinite amenable then $\Xi(G_1^{\alpha_1}) = \Psi(G_1^{\alpha_1})$ and therefore $\hat{\Xi}(G^\alpha) = 1 - n + \Xi(G_1^{\alpha_1}) = n_1 - n$.

(v) When G_1 is finite then $\zeta(\mathcal{H}_1')$, $\mathcal{H}_1' \in \mathcal{CF}(\mathcal{G}_1)$ achieves its maximum on $\mathcal{G}_1$.

(vi) When G_i is cyclic then $\Xi(G_i^{\alpha_i}) = (|G_i|)^{-1}$ and $\Psi(G_i^{\alpha_i}) = (|G_i| - 1)^{-1}$. Then when $G_{1,2}^\alpha$ is the induced presentation of $G_1 * G_2$ we get from Theorem 5.3 that

$$\begin{aligned} \hat{\Xi}(G^\alpha) = 1 - r + \Xi(G_{1,2}^\alpha) &= 1 - r + \frac{1}{|G_1|} + \frac{1}{(|G_2| - 1)|G_1|} \\ &= 1 - r + \frac{1}{|G_1|(1 - \frac{1}{|G_2|})}. \end{aligned} \tag{90}$$

(vii) This is also clear by (67). ∎

We see from Corollary 5.5 (iii) that if G is the free product of two non-trivial finitely generated groups G_1 and G_2 then G is amenable if and only if both G_1 and G_2 are cyclic of order 2, because $\hat{\Xi}(G^\alpha) \leq 1 - 2 + \hat{\Xi}(G_1^{\alpha_1}) + \hat{\Xi}(G_2^{\alpha_2})$ and $\hat{\Xi}(G^\alpha) = 0$ if and only if $\hat{\Xi}(G_1^{\alpha_1}) = \hat{\Xi}(G_2^{\alpha_2}) = 1/2$. In fact, this is well known, and in this case G is the infinite dihedral group which is the semidirect product of an infinite cyclic group and a cyclic group of order 2, and the fact that G is then amenable follows also by Corollary 3.2.

6. The normalized balanced cyclomatic quotient

The definition we used for $\hat{\Xi}(G^\alpha)$ says that its value has to be looked for in the "best chain" we can find in the graph. If we look on the other hand only on the chain which consists of the concentric balls around 1, we get some "averaging" and in this case the value we calculate depends on the growth of the group. Indeed, the number of vertices in a ball of radius i in the Cayley graph equals $\Gamma(i)$, the number of group elements whose length (with respect to the presentation) is at most i.

Definition 6.1: Let G^α be n-generated and let $\mathcal{G}$ be the corresponding Cayley graph. Let $\mathcal{B}_i$ be the (induced) concentric balls around 1 of radius i in $\mathcal{G}$. Then we define the **normalized balanced cyclomatic quotient** of G^α by

$$\hat{\Theta}(G^\alpha) = 1 - n + \limsup_{i \to \infty} \xi(\mathcal{B}_i). \tag{91}$$

Clearly $\hat{\Theta}(G^\alpha) \leq \hat{\Xi}(G^\alpha)$, and equality holds when G has sub-exponential growth (i.e. when the growth function $\Gamma(i)$ of G is not bounded below by some exponential function, or in other words $\lim_{i\to\infty}\Gamma(i)^{1/i} = 1$), as seen from the proposition below. When G is infinite then $\hat{\Theta}(G^\alpha)$ is between $1-n$ and 0.

PROPOSITION 6.2: *The growth of G is exponential if and only if $\hat{\Theta}(G^\alpha) < 0$.*

Proof: This follows by having $1 - n + \xi(\mathcal{B}_i) = (1 - |E^X_{out}(\mathcal{B}_i)|)/\beta_0(\mathcal{B}_i)$, and $\beta_0(\mathcal{B}_i)$, $i = 0, 1, 2, \ldots$ is (a representative of the equivalent class of) the growth function of G. ∎

We say that a non-empty subgraph $\mathcal{H} \subseteq \mathcal{G}$ has **thickness** $\geq r$ if $\mathcal{H}$ contains a non-empty subgraph $\mathcal{H}'$ such that $d(v, \mathcal{H}') = r$ for every $v \in \partial\mathcal{H}$. The supremum on all such r is the thickness of $\mathcal{H}$. When $\partial\mathcal{H}$ is empty then the thickness of $\mathcal{H}$ is ∞. Thus every subgraph has thickness ≥ 0, and it has thickness ≥ 1 if and only if every vertex on its boundary is adjacent to an interior vertex.

We call a subgroup $H < F$ a **supnormal** subgroup if the maximal normal subgroup N of F which is contained in H is non-trivial. In the next lemma we refer to the length of a shortest word in N, which is the girth of the Cayley graph of F/H in case H is normal.

LEMMA 6.3: *Let H be a supnormal subgroup of the free group F of rank n and let $m > 0$ be the length of a shortest word in the maximal normal subgroup of F contained in H. Then for every finite subgraph $\mathcal{G}'$ of the cosets graph of H with thickness $> m/2$*

$$\xi(\mathcal{G}') \geq \frac{2(n-1)}{m((2n-1)^{1+m/2} - 1)}. \tag{92}$$

Proof: Clearly it is enough to show the claim holds for connected subgraphs. Let λ be a fixed circuit of length m corresponding to an element as in the lemma. Let $\mathcal{G}'$ be a connected finite subgraph of thickness $\geq m/2$ in the cosets graph of H, and let $\mathcal{G}''$ be the subgraph of $\mathcal{G}'$ induced in $\mathcal{G}'$ by the set of vertices of distance $\geq m/2$ from $\partial\mathcal{G}'$. We cover $\mathcal{G}''$ with copies of λ, so that each new circuit begins in a vertex not covered yet. Part of the circuits may extend beyond $\mathcal{G}''$, but not beyond $\mathcal{G}'$. Then

$$\beta_2(\mathcal{G}') \geq \frac{1}{m}\beta_0(\mathcal{G}'') \tag{93}$$

since each circuit was counted at most m times. As for the cardinality of $\mathcal{G}'$, we have

$$\beta_0(\mathcal{G}') \leq \beta_0(\mathcal{G}'') + a\beta_0(\partial\mathcal{G}'') \leq (1+a)\beta_0(\mathcal{G}''), \tag{94}$$

where

$$a = \sum_{j=1}^{m/2}(2n-1)^j. \tag{95}$$

Therefore

$$\xi(\mathcal{G}') \geq \frac{\beta_0(\mathcal{G}'')}{m(1+a)\beta_0(\mathcal{G}'')} = \frac{2(n-1)}{m((2n-1)^{1+m/2}-1)}. \tag{96}$$ ∎

As an immediate corollary we have

PROPOSITION 6.4: *Let $G^\alpha = \langle X | \, R\rangle$ be a presentation of a non-free group G with $|X| = n$. Let m be the length of a shortest relator. Then*

$$\hat{\Theta}(G^\alpha) \geq 1 - n + \frac{2(n-1)}{m((2n-1)^{1+m/2}-1)}. \tag{97}$$

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